CEMC GRADES 9/10 MATH CIRCLES
NOVEMBER 23/30, 2022
FORMAL LOGIC - SOLUTIONS

1. Introduction

Find the error in the following arguments.

(1) I have only ever seen black cats. Therefore all cats must be black.

(1) Just because I haven't seen a white cat doesn't mean they don't exist.

(2) Sharks eat fish. I am not a fish. Therefore a shark will not eat me.

(2) We don't know that sharks *only* eat fish.

(3) If I eat healthy and exercise I will become stronger. I eat healthy. Therefore I will become stronger.

(3) You need to both eat healthy *and* exercise to be guaranteed results.

(4) All stock brokers want to make money. Smarter investors will make more money. Money does not buy happiness. Therefore smart stock brokers will not be happy.

(4) Smart investors will make more money and that is not *guaranteed* to make them happier, but it might.

2. Implication

Transform the following statements into an implication using "If... then...".

(1) You'll catch a cold without a coat!

(1) If you don't wear your coat then you will catch a cold

(2) Buy one get one free.

(2) If you buy one then you will get one free

(3) I get sleepy when I read.

(3) If I read then I will get sleepy.

(4) Beautiful sunsets deserve to be seen.

(4) If a sunset is beautiful then it deserves to be seen.

What is the hypothesis and conclusion in the following implications?

(1) If she stands too close to the edge then she will fall.

(1) Hypothesis: She stands too close to the edge. Conclusion: She will fall.

(2) If everyone wants to go to the mall then we can go.

(2) Everyone wants to go to the mall. Conclusion: We can go.

3. Other Logical Connectives

Let $P =$"I sing loudly", $Q =$"I dance well" $R =$"The audience claps". Write the following logical statements in English.

(1) $P \wedge R$ "I sing loudly and the audience claps"

(2) $P \Rightarrow R$ "If I sing loudly then the audience claps".

(3) $(P \vee Q) \Rightarrow R$ "If I sing loudly or dance well then the audience claps".

(4) $(\neg R \Rightarrow \neg Q)$ "If the audience does not clap then I do not dance well"

Fill in the following truth tables

(1)

P	$P \vee \neg P$
T	T
F	T

(2)

Q	$Q \wedge \neg Q$
T	F
F	F

(3)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(4)

P	$\neg P$	$\neg\neg P$
T	F	T
F	T	F

4. Tautologies and how to win arguments

Are the following formulas tautologies?

(1) $P \Rightarrow (P \wedge Q)$

(1) No. Consider the following partial truth table

P	Q	$P \wedge Q$	$P \Rightarrow (P \wedge Q)$
T	F	F	F

(2) $\neg\neg P \Rightarrow P$

(2) This is a tautology. It has the truth table

P	$\neg\neg P$	$\neg\neg P \Rightarrow P$
T	T	T
F	F	T

(3) $(P \vee \neg P) \Rightarrow Q$

(3) No. Consider the following partial truth table

P	Q	$P \vee \neg P$	$(P \vee \neg P) \Rightarrow Q$
T	F	T	F

(4) $(P \wedge \neg P) \Rightarrow Q$

(4) This is a tautology. It has the truth table

P	Q	$P \wedge \neg P$	$(P \wedge \neg P) \Rightarrow Q$
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

Check if the following arguments are valid or not.

(1) If you give a mouse a cookie then he's going to want some milk.

(2) If you give a mouse some milk he's going to want a straw.

(3) Therefore if you give a mouse a cookie he's going to want a straw.

This is a valid argument. Let $P =$"You give a mouse a cookie", $Q =$"he wants some milk", and $R =$"he wants a straw". Then our hypotheses are $P \Rightarrow Q$ and $Q \Rightarrow R$. Our conclusion is $P \Rightarrow R$. Our argument is valid if $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology. We have the following truth table.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

(1) If you're still doing these problems then you must like math

(2) Therefore if you do not like math you are not still doing these problems

This is a valid argument. Let $P =$"You're still doing these problems" and $Q =$"You like math". Then our only hypothesis is $P \Rightarrow Q$ and our conclusion is $\neg Q \Rightarrow \neg P$. our argument is valid if $(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$ is a tautology. We have the following truth table.

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$	$(P \Rightarrow Q) \Rightarrow (\neg Q \Rightarrow \neg P)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

(1) A person is tall or short

(2) A short person is rich or poor

(3) Therefore a person is tall or rich or poor

This is a valid argument. Let $P =$"A person is tall", $Q =$"a person is short", $R =$"A person is poor", and $S =$"A person is rich". Then our hypotheses are

$P \vee Q$ and $Q \Rightarrow (R \vee S)$. Our conclusion is $P \vee (R \vee S)$. Our argument is valid if $((P \vee Q) \wedge (Q \Rightarrow (R \vee S))) \Rightarrow (P \vee (R \vee S))$ is a tautology. We have the following truth table (which is too big for one line).

P	Q	R	S	$P \vee Q$	$Q \Rightarrow (R \vee S)$	$((P \vee Q) \wedge (Q \Rightarrow (R \vee S)))$	$(P \vee (R \vee S))$
T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T
T	F	T	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	T	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	T	F	T	F	T
F	F	T	F	F	T	F	T
F	F	F	T	F	T	F	T
F	F	F	F	F	T	F	F

$((P \vee Q) \wedge (R \vee S)) \Rightarrow (P \vee (R \vee S))$
T
T
T
T
T
T
T
T
T
T
T
T
T
T
T
T

5. Semantics

Which of the following logical equivalences are correct?

(1) $(P \vee \neg P) \vee Q \equiv Q$
(2) $(P \wedge \neg P) \vee Q \equiv Q$
(3) $(P \vee \neg P) \wedge Q \equiv Q$
(4) $(P \wedge \neg P) \wedge Q \equiv Q$

(Answer) 2 & 3

Notice that $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$, and so we can write it without brackets as their is no confusion in saying $P \vee Q \vee R$.
WITHOUT using a truth table show that $\neg(P \vee Q \vee R) \equiv \neg P \wedge \neg Q \wedge \neg R$.

By one application of our rule to $\neg((P \vee Q) \vee R)$ we get $\neg(P \vee Q) \wedge \neg R$. By applying it again we get $\neg P \wedge \neg Q \wedge \neg R$.

6. Contrapositive

Prove the following statements by contrapositive.

(1) If $x = 2$ then $3x - 5 \neq 10$.

(1) The contrapositive of the statement is "If $3x - 5 = 10$ then $x \neq 2$". If we assume that $3x - 5 = 10$ then it follows that $x = 5$ so it is true that $x \neq 2$.

(2) If x, y are two integers such that $x + y$ is even, then x and y have the same parity (both odd or both even).

(2) We assume the negation of the conclusion, that is, that x and y have different parity. and so we may assume that x is even and y is odd (otherwise just swap the letters). The sum of an even number and an odd number is odd, so $x + y$ is not even. This is the negation of the hypothesis, so the statement has been proven.

7. Reductio ad absurdum

Prove the following statements by contradiction.

(1) If x^2 is even then x is even.

(1) Assume that x^2 is even AND that x is not even. Then since x is an integer it is odd. But the product of two odd integers is odd, so x^2 must be odd. Therefore x^2 is odd AND it is even. This cannot be, so if x^2 is even then x must be even.

(2) If $x, y \in \mathbb{R}$ are such that $x > 0$ and $y > 0$ then if $xy > 25$ at least one of x, y must be greater than 5.

(2) We assume towards a contradiction that $xy > 25$ and that x, y are both less than or equal to five. But if each number of between 0 and 5 then the biggest their product can be is 25. So $xy \leqslant 25$ and $xy > 25$. This is a contradiction, and so the original implication is proven.

(3) The square root of an irrational number is irrational.(A number is rational if it can be written as a fraction $\frac{p}{q}$ where p, q are integers.

(3) Assume that x is irrational and that $\sqrt{x}$ is rational. Then since $\sqrt{x}$ is rational we can find some p, q such that $\sqrt{x} = \frac{p}{q}$. But

$$x = (\sqrt{x})^2 = (\frac{p}{q})^2 = \frac{p^2}{q^2}.$$

This is a contradiction, because we have assumed that x is not rational and we have written it as a fraction where p^2, q^2 are integers. This proves the original statement.